The Mathematics Teacher

DEVOTED TO THE INTERESTS OF MATHEMATICS
IN JUNIOR AND SENIOR HIGH SCHOOLS

VOLUME XVII DECEMBER, 1924 NUMBER 8

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The Mathematics Teacher

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THE MATHEMATICS TEACHER

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THE GENERAL TREND OF MATHEMATICS EDUCATION IN SECONDARY SCHOOLS

By W. D. REEVE
Teachers College

The most significant thing in education today is the wide recognition of individual differences in ability among pupils. This is particularly true in regard to the high school and is due partly to a realization that our secondary school population is very different from that of thirty years ago "not only in their experiences and interests, but also in their inborn abilities." According to a recent report of the division of research of the National Educational Association,[1] the number of pupils enrolled in high schools in this country has increased from 202,963 in 1890 to 2,229,407 in 1922. In this report we read that "If the population of the United States had increased as rapidly as its high school enrollment since 1890, its general population would now be 687,861,591."

Thus, we now have a high school population that is not only very different from that of thirty years ago, but it is also much larger. Many of those who have daily contact with the general run of high school pupils declare that the pupils of today as a group are also inferior to those of a short time ago. Nevertheless, as Thorndike[2] points out, there are in the high school group a "few (less than 3 per cent.) who have not median intelligence (a score of 65) or better." (The score of 65 refers to the Army Alpha test.) He says further, that "The pupils in academic high schools are, in fact, a limited group which cover just about half, the upper half, of the total distribution of American intelligence."

In spite of the fact that the high school group is limited, any thoughtful teacher knows that within the group one always

[1] The *Journal* of the National Educational Association XIII, 9 p. 773.
[2] Thorndike, E. L., *Psychology of Algebra,* p. 24.

finds a wide range of abilities. In fact the range is so great in some classes that one cannot teach the most brilliant pupils along with the slowest ones without doing an intense injustice to both. It is probably true that one cannot really teach them together at all in the best sense of the word.

This sad state of affairs has led in many of the larger schools to homogeneous grouping according to ability. I have used this plan for several years and can say that its advantages far outweigh any criticisms that may be brought against it. The most common charge that homogeneous grouping is not democratic is greatly offset by actual results because in a democratic situation the most important function of the school is the training of leaders.

We shall be able in certain schools to classify pupils in homogeneous groups; but in many schools, especially the smaller ones, this will be impossible. This situation presents a problem which challenges the best thinking of the most highly trained and experienced teachers of mathematics. In the end, it means different amounts of possible material for the varying degrees of ability, if indeed it does not mean different courses altogether in so far as quantity and quality are concerned. As a result, we shall doubtless have a certain so-called course of "minimum essentials" beyond which we shall add material that will enrich the course for those who are able to do more intensive and extensive work. The other alternative will be to permit pupils to progress as rapidly as possible. However, the latter plan has certain social disadvantages that do not commend it to many people who have seen it work.

Someone has recently said, "We must now know as a result of our knowledge of individual differences of ability that our present practice of teaching such varying types can only lead to a sort of educational bankruptcy. It is inevitable, therefore, that certain adjustments must be made. Within reasonable limits we must abandon the ironclad traditions governing group instruction."

It now seems clear that with reference to algebra in particular, the pupils who elect that subject "Are in general a more intelligent group than those who do not; pupils who pass in algebra are in general a more intelligent group than

those who take it and fail." Moreover, it has been suggested that in general, "a pupil whose first trial Alpha ability is below 100 will be unable to understand the symbolism, generalizations and proofs of algebra" even though he may receive credit for passing the course. Whether the proper figure is 100, 105, or something else is not important, but if it is true that there exists some point below which the pupils who are found there cannot profit by taking the ordinary course in algebra, then everybody concerned should know it. The important thing for us to remember is that as a general rule the less intelligent a pupil is the less will be his gain from a year's study of algebra.

Again, we have had too many failures in algebra, and geometry in the past. It does a pupil no good to fail and yet we find schools even today where the failures are as high as 25 and 30 per cent. for a normal group. This procedure can not be defended unless the situation is unusual.

The preceding discussion raises two important questions. First, the question of the wisdom of requiring a year of algebra of all pupils to say nothing of a year of geometry, and second, the question as to the best way of dealing with the differences in ability among the pupils who are able to take the course. It appears to me that the question as to whether a certain course in mathematics should be required of all pupils can be answered by a scientific determination of the amount of achievement that can actually be made in a given course. However, we should be sure that in such cases the teaching has been of a high order. If it turns out that there are some pupils who cannot learn mathematics, or who profit to a very small degree by taking it, then I should not require it of them. If I had my way, I should require mathematics only in the junior high school. The joy that would come to teachers and pupils alike by having in the tenth, eleventh, and twelfth year classes only the pupils who love mathematics so that they take it by choice would be a great blessing to all concerned. Moreover, I am confident that we mathematics teachers would still have classes to teach.

Naturally, the question of what to do with pupils who are excused from certain courses will arise. If they are to take

mathematics at all, they should be given a course with which they can succeed. Such a course can be worked out and has been worked out in some schools by experiment. I believe that some modified form of general mathematics would meet the needs and interests of such pupils. Some schools put the algebra course in the tenth or eleventh grade because of its difficulty for certain pupils, but I doubt the wisdom of such a plan. In any event, it seems to me that we ought to face the situation squarely. We should not fail to reorganize subject matter and method; because if we do, we shall reach the place where school officials will not want to require algebra of anybody.

A proper reorganization to meet the varying needs of pupils will require careful experimentation in places where teachers are qualified to do the work. Besides, the percent of pupils who cannot learn algebra will vary in different localities and schools. We cannot cut off a certain percent and let it go at that. We shall have to find out through experience just what it is that pupils can learn and the most economical methods of learning it. We shall assume of course that in such a scheme nothing will be taught by the teacher that is not worth learning by the pupil. The worthwhileness of the material learned will have to be judged by criteria which we must set up. This raises the whole question of purposes, or objectives, and, ideally, that is where the organization of our course of study in mathematics should begin. Thus, we must decide first, as far as we can, what we hope to do for our pupils, and what the purpose of the various parts of the course is. Second, we must decide upon the content best suited to our purpose; and third, we must find out how to teach the content most efficiently and economically. This necessitates a careful study of the learning process and involves an elaborate testing program so organized as to be truly diagnostic, and such that when weaknesses are found, they may be met with proper remedial measures.

No matter what plan is adopted for the inferior pupils we can profit by its use, inasmuch as we may use what we learn in overcoming the special disabilities of slower pupils in the instruction of more superior ones. We want such a scheme to result in more attention to the needs of individual pupils, less

failures in every group, the elimination of waste in general, and a greater joy in the life of the classroom.

The final report of the National Committee on the Reorganization of Mathematics has been and will continue to be of great assistance to teachers all over the country in planning modern courses of study. It is to be regretted, however, that the opinions of teachers in allied fields as given in chapter 5 on the value of various parts of mathematics is not more valuable in showing how our subject might be used there. Does the fact that a group of teachers in an allied field fail to list a certain topic in mathematics as valuable to them mean that mathematics is or can be of no service there? Is it not possible that there are large uses of mathematics if the teachers in these fields only knew how to use it? I do not believe that we can depend upon ratings given in such surveys by teachers who are often not familiar with many of the possible uses of mathematics. In some fields, at least, it is probable that the interest of pupils would be increased and the subject enriched by the introduction of mathematics now unknown to these fields. I am thinking particularly of the wise use that might be made of graphs and formulas if only the teachers of certain subjects knew that such things existed.

One of my friends remarked in this connection the other day that the tooth brush was still an important instrument for children to use even though many do not think of it often. And so it is with algebra.

I am inclined to think that if the teachers in some of these fields are ever to admit that mathematics is of use to them, we shall have to show them where such use can be made. One thing is certain and that is that there is general agreement among those who use mathemtics as to the wisdom of giving greater emphasis and time to the formula, the equation, directed numbers, statistics and graphic representation in general. However, these topics are not generally taught in high school algebra classes in this country.

In some respects at least every teacher should be a teacher of mathematics just as every teacher should be a teacher of English. Even though people generally may not realize it, our pupils who aspire to be leaders must know algebra and its

language whether they expect to compute with it or not. They will be compelled to know mathematics in order to understand much that will want to read in their daily lives. Thus, we shall have to decide what must be required of all pupils in the various years. What is required will be determined by the criteria which are set up and it is to be hoped that these criteria will be carefully selected and checked before any final decision is made.

One thing seems sure to follow if such a plan is honestly carried out, and that is, that teachers of mathematics themselves will give more serious study to the best types of teaching material and methods of presentation. This will mean further attendance at summer sessions, reading new books, and so on. Moreover, writers of textbooks will be compelled to give greater attention to the knowledge and skill that is necessary to present material in more desirable and useful ways. The entire matter of psychological order of subject matter will also be involved in any reorganization program. The planning of a good course of study in mathematics necessitates also a more complete study and tabulation of the nature and meaning of mathematical abilities and terms of various kinds.

We have doubtless overemphasized the value of problem solving in the past. It is possible that the right kind of thinking about some of the more formal processes may be as conducive to power in mathematics as is the solution of problems. A newer tendency in this connection is to have pupils write the equations expressing the conditions of various problems, but not to solve all of them. The ability to solve formal equations can be tested and is usually tested elsewhere.

One of our biggest problems in making a course of study is to decide how much consideration and time shall be given to certain topics, how difficult the task within these topics shall be, and what real applications can be made of them.

As has been hinted earlier, teaching is only one half of the classroom situation and it is not the most important half. The way the pupil learns and the most economical methods of learning have in the past been given too little attention. We need to make an honest application of the laws of learning to classroom practice. The difficulty that pupils often have in

learning a topic is due more to the clumsy habits of the teacher than to his own inability to learn. The fact is that teachers often get in the way and I should not want to stamp a pupil as unable to learn mathematics unless I was sure he had been given a fair chance. This chance must be given to all pupils, and the place to give it is in the junior high school.

We need to remember also that definitions should be the outgrowth of the work rather than the basis for it. We must find out, if we have not learned it before, that pupils come to know things better by working with them rather than by studying what is said about them. We all know from experience that pupils learn best what a topic means by solving the tasks connected with it. The pupil who comes up short of a reasonable achievement in this respect is usually lacking in his knowledge of the topic.

We should emphasize also the need of basing our teaching of new topics upon the past experience of the pupils and to modify what they have already learned in order to give them new experiences that will fasten things in their minds.

One would hardly need the advice of a psychologist to discover that the teaching of mathematics would be easier if we could get rid of much of the deadwood, unfortunate terms, and symbols. If we could do this, we could make room for many desirable additions that would give us a richer, broader, and a more enjoyable course of study. Consider, for example, the old $a:b::c:d$ method of writing a proportion; the words "antecedent," "consequent," "composition," "division," "comparison," and so on. A careless and often meaningless use is made of the words "transpose," "cancel," "clear of fractions," and the like. David Eugene Smith has pointed out that few people ever stop to think what "minuend" and "subtrahend" mean, if they know their meaning at all. A blacklist of all such terms should be made and they should be discarded by all of us as soon as possible. Then and only then can we make our first courses such that pupils, who ought to continue the study of mathematics, will be led to do so, and such that they will influence school authorities to provide instruction in the later years for those who love mathematics and who may profit by studying it further.

In other words, I should say "teach the mathematics that ought to be taught, in the way it should be taught and it will not have to be justified by the doctrine of transfer." Moreover, the pupils who get a real acquaintance with the subject will respect and love it rather than hate it as they often do.

We are often advised by psychologists to avoid "superfluous bonds." For example, shall the formula method of solving a quadratic equation be used exclusively in view of the fact that a large percentage of quadratic equations that life may offer are not factorable? It is certain that some outstanding mathematicians will not agree. Who is right or nearest right? We ought to know whether it is worth while to teach pupils to factor quadratics at all.

I visited a class recently where a pupil insisted on solving an exercise in an algebra text by arithmetic while his teacher struggled to get the algebraic method from him. Textbooks and teachers should not require an algebraic solution of a problem or exercise which is better and perhaps more easily solved by arithmetic. If such care is not exhibited by textbooks and teachers, then it may be true, as has been charged that "algebra is arithmetic done in a harder way." There is an abundance of good algebra material that cannot be handled by arithmetic and we ought to know what it is.

We need to set up reasonable objectives subject to change with new knowledge that will accumulate. We ought to make a course of study that will best help us to realize these objectives. We need to develop new types of exercises in algebra, and to make a better use of tests in finding out what success our pupils have. We need also to change our emphasis on certain topics. If we teach quadratic functions at all, then we should give more emphasis to a consideration of $y=ax^2+bx+c$ for the whole family of values of y and less to the very special case where $y=0$. A similar argument will apply to the case, $y=ax+b$.

Again, where does the emphasis need to be shifted? Is factoring, for example, of as much use as its traditional prominence indicates? A similar question may be raised with reference to simultaneous equations which practically all pupils enjoy solving, but the value of which is almost negligible for most of

them. We may ask ourselves the question "Shall we continue to teach topics which the children enjoy, but which are not as valuable as other possible material may be?" Much of the work on the formula might be harder, but it is also far more important and, in the long run, better for the pupils to study.

Has the ordinary textbook too much, too little, or the wrong kind of drill material? Is it quality of drill that is needed most, or is the lack of quantity the greatest handicap?

Should everything be completely mastered? If not, what things need not be, and what degree of mastery should we require? Personally, I believe that not all things in algebra need be, or should be, completely mastered. It depends entirely upon the purpose to be served. The fact is that nobody knows how much drill it takes to teach anything. How long, for example, would it take a normal group of first year pupils to learn to add directed numbers with a certain degree of proficiency?

What is the most valuable kind of practice exercise? Perhaps it is not necessary to know the answer, but it is rather important to set up certain standard ideals of achievements for any group.

Should the study of directed numbers come before, with, or after the first lessons on the formula and the equation? One often finds differences of opinion. Is there a best way to teach anything? Perhaps not because teaching is an art. However, is it not possible that we may discover many better ways of doing some things if we try?

Why is it that when pupils attempt to solve the equation—

$\frac{x}{2} + \frac{x}{3} = 8$, they get $3x + 2x = 8$, but in adding $\frac{x}{2}$ and $\frac{x}{3}$ they usually get $\frac{3x + 5x}{6}$ or something else? These are all questions that should be answered by all thoughtful teachers.

We might ask further when and where should arithmetic, algebra, and geometry be introduced into the junior high school? Would it be better to begin the junior high school course with intuitive geometry than with arithmetic? Of course there are arguments on both sides. Is one way any

better than the other, or is it of no importance to ask? We ought to know. School authorities want to know whether present methods can be improved.

Any failure on the part of teachers of mathematics to be awake as to the needs of modern life is sure to be followed by a refusal of state and municipal authorities to require any mathematics at all beyond the eighth grade. Some people are already saying that one year is all we can spare for plane and solid geometry. I am inclined to agree with that attitude. It is the only way we can cut the "ancestral process" short and give our pupils some of the more advanced mathematics that will be of greater value and interest. If we omit the proofs in plane geometry which are based on limits as most of the modern texts are now doing and include only the fundamental theorems of plane and solid geometry, together with a wise choice of applications, we shall provide the pupil with a more enjoyable course and give him just as sound a basis for future work as we have given heretofore.

With similar reorganization in the more advanced parts of of algebra and trigonometry, time will be gained which can be spent on the fundamentals of analytic geometry and calculus for those who elect to go on in the study of mathematics.

We must at least give serious consideration to the demands of the public for a live, throbbing course and take the lead ourselves in the right kind of reorganization if mathematics is to hold the high place it deserves in the modern high school curriculum.

THE RECITATION IN MATHEMATICS[1]

By EDITH IRENE ATKIN
Illinois State Normal University, Normal, Ill.

Many movements in education are claiming the attention of teachers of mathematics. Some teachers and administrators would make all high school mathematics elective, some are teaching some form of general mathematics, some are emphasizing standardized tests, some have instituted supervised study or instruction by homogeneous groups, and others are working on the content of courses in mathematics for both the junior and the senior high schools. These different movements all indicate an earnest desire on the part of their exponents to help in the solution of educational problems, and every progressive teacher is vitaily interested in the results, either from the standpoint of a participator or an observer.

But is it not true that the large majority of high school teachers are teaching the usual prescribed and elective courses in algebra and geometry in forty or forty-five minute periods from textbooks, which reflect in varying degrees modern trends? As members of this large group, may it not be well for us to take a survey of certain fundamental principles which should guide mathematics teachers in their work?

I have read the primary purposes to be realized in the teaching of mathematics as laid down by the National Committee on Mathematical Requirements, and the purposes as stated by many writers of textbooks and other educators, and finally, I have formulated the aims which I shall assume represent current thought. The purposes of the teaching of mathematics are:

(1) To impart a knowledge of those facts, processes, and applications of mathematics which serve as a minimum requirement in adequately controlling our environment, which will function in other high school subjects, and which will at the same time, prepare for the future study of mathematics.

(2) To develop an appreciation of the beauties of mathematics, and an appreciation of the power of mathematics in the world.

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[1] A revised copy of a paper read before the Mathematics Section of the High School Conference, University of Illinois, Urbana, November 24, 1922.

(3) To develop those powers of thought and action which will make a good citizen.

The classroom teacher meets his class perhaps one hundred eighty times during a year in periods of forty-five minutes each. In these one hundred eighty contacts covering one hundred thirty-five hours the teacher must make his impressions, and at least partially realize these purposes. In this discussion we shall accept the content of the courses as given by our later text books, and center our interest around the recitation.

The four lines of thought to which I wish to call your attention are:

(1) Every pupil in the class should feel that mathematics is worth studying.

(2) The recitation should be conducted according to the principles of scientific business management.

(3) The dominant aim governing classroom procedure should be to keep every pupil thinking on the lesson all the time.

(4) The ultimate aim should be to produce in our pupils scholarly attitudes of mind, and make self-reliant workers.

By no means are these mutually exclusive ideas, but they will serve as thoughts to guide our discussion.

Every pupil in the class must feel that mathematics is worth studying. This emotional response is essential to effective work. Psychologists tell us that educative acts must be purposive acts. If the pupil feels that mathematics is just "stuff," that it is of no use, that it is hard and that it will be impossible for him to understand it, little can be accomplished.

In overcoming these reactions, the teacher is a most important element. The recent great German mathematician, Felix Klein, said: "If I were to formulate the great problem of pedagogy mathematically, I should say that it consisted in marshaling the individual qualities and capacities of the teacher and his students as so many known variables and in seeking the maximum value for a function of $(1+n)$ variables, $F(x_0, x_1, \ldots x_n)$, under given collateral conditions. If this problem as a result of the advances made by psychology should some day admit of direct mathematical solution, then from that day onward practical pedagogy would have become a science. Until then it must remain an art." The teacher's personality, his scholarship, his

enthusiasm for his subject, his sympathy, his open-mindedness, are dominant factors in all successful instruction.

The direct practical uses which we can lead the pupil to make of the equation in algebra or the theorem in geometry are limited, but we must make him see that he is enjoying the results of the uses to which other people have put mathematics. Lead him on an excursion into the unseen, and help him to discover that there is mathematics back of the house he lives in, the clothes he wears, the food he eats, the books and newspapers he reads, the artificial light he uses, the street car or automobile he rides in, the bridges he crosses, the buildings he enters.

When the class are studying some particular geometrical form, ask them to look for this form in the school yard, city, and country. Pupils will discover triangles in shapes of leaves, in shapes of trees, in gable ends of buildings, in sections of roofs, in supports for brackets, in scaffolding, in saw horses, in derricks, in gates, and in bridges. They can be led to appreciate geometric design in the honeycomb, spider web, birds' nests, snowflakes, in primitive shelters, in Indian basketry, pottery, and bead work, in tiled floors, in ceiling and wall decorations, in church windows, etc. In connection with these observations, such questions as the following are pertinent. Why is the triangle used as a brace? Why are not regular pentagons used as designs for tiled floors? Why is the six-sided cell the best form for the honeycomb? Reproduce this design for a church window.

The students in the elective courses will be interested in knowing that the actuary in the office of the life insurance company uses geometrical series, the binomial theorem, and the theory of probability in computing premiums, and that economists, experts in finance, and social investigators should know the theory of investment and the theory of statistics—subjects which rest upon a mathematical sub-structure.

When the pupil makes graphs of quadratic equations, he learns that the curve $y^2 = 4ax$ is called a parabola. How his interest in this curve is awakened when he knows that it appears in the flight of a ball through the air, in the automobile headlight, in lighthouse reflectors, in mirrors for focussing the sun's rays, and in theatre and auditorium construction. Show him a cone and point out the different conic sections, and tell him that it was

nineteen hundred years after Menaechmus commenced his study of conic sections before Kepler discovered that the planets move in elliptical orbits. Interest him in the recent discovery that all comets move in elliptical orbits, though sometimes very elongated, as opposed to the old idea that those which return at regular intervals, like Halley's comet, move in ellipses, and those which have been seen but once come from without our solar system in parabolic or hyperbolic paths. Tell him the story of the two young men, Adams of England and Leverrier of France, who discovered a new world by mathematics. Astronomers had discovered that Uranus was wandering out of the path expected of it and behaving irregularly. What was the cause? Adams and Leverrier set out independently to solve the problem, and computed these disturbances to be due to the attractive force of another planet the very existence of which was unknown. They determined the orbit of this planet, and told the astronomers that if they would turn their telescopes to a certain part of the heavens at a certain time they would there find a new world. It was found by Galle at Berlin, September 23, 1846, and named Neptune.

One type of the beauty of mathematics appears in the harmony and interplay between algebra and geometry. The graphs of equations show this admirably. The square of the sum of two numbers may be expressed algebraically and geometrically, the segments of a line divided in extreme and mean ratio may be found by compasses and straight edge, and also computed by algebra. The pupil may have started out to find only the segments made by the internal division, but algebra is generous, and gives him both the segments of the internal and of the external division with one asking. Psychologists tell us that the rectangle most pleasing to the human eye is that in which the sum of the two dimensions is to the longer as the longer is to the shorter. Thus we see this Golden Section of the Greeks applied to modern art.

Is there not a beauty in the thought that the truths of mathematics are eternal? Chemistry, physics, and physiology texts written twenty-five years ago contain many statements now known to be false, but the truths of mathematics compiled by Euclid three hundred years before Christ are just as true today as they were then.

In the mathematical formula, the pupil should see a shorthand expression of a law, and in the equation, a great labor saving device for the solution of problems. As many different countries use practically the same symbolism, mathematics has been called the language of languages. Do we as teachers appreciate that the mathematical formula is the ultimate form in which truth in all subjects seeks to express itself?

I have been much interested in a recent illustration of this fact from a field not generally supposed to be adapted to mathematical investigation—medicine. In 1908 and again at the outbreak of the war, Dr. Alexis Carrell of the Rockefeller Institute of Medical Research noticed that the rate of healing of a wound seemed to be proportional to its surface area. He plotted curves using the time as abcissa and area of wound as ordinate, and came to the conclusion that the relationship could be represented by an equation. He turned over the matter to Dr. du Noüy of one of the base hospitals in France who developed the formula,

$S_n = S_{n-1}[1 - i(4 + \sqrt{4n})]$ where

S_n is the area after $4n$ days,

S_{n-1} is the area after $4(n-1)$ days, and

$i = \dfrac{S_1 - S_2}{6S_1}$ where S_1

is the measure of the area on first observation, and S_2 a second measurement taken after four days. The surface area of the wound is traced on paper, and then computed on a mathematical machine called the planimeter.

If the plotted observation does not agree with the graph of the formula, it means that there is something wrong with the healing, such as infection in the wound. In this way infection can be found out by mathematics before its presence could be detected by medical examination. The formula, too, will enable the research worker in medicine to determine the action of different fluids on the healing of a wound. (The data given are taken from Karpinski, Benedict and Calhoun's Unified Mathematics, a text book for college freshmen.)

Pupils may be led to a better appreciation of mathematics by frequent references to its history. It is most unfortunate to have pupils think, if they think about it at all, that algebra and geometry were found somewhere ready made. Some knowledge of the history of a topic has a most stimulating effect upon the pupil's interest. A study of the history is most valuable, too, for the teacher for it gives him a background from which he can critically evaluate current movements, and it makes him a better teacher for the way the race has learned its mathematics is indicative of the way the child will learn.

Also I want the pupil to know that mathematics is still in the making. In his address in 1920 before the mathematical section of the American Association for the Advancement of Science, on the topic, "A Decade of American Mathematics," Prof. O. D. Kellogg of the University of Missouri discusses the contributions made to mathematics during the decade from 1910 to 1920. By his investigation he found that three hundred twenty-five persons had contributed one thousand two hundred fifty-eight articles in this interval. The research work of tomorrow must be done by somebody's pupils of today.

When would I try to teach all this? I would have periods, perhaps one a month, devoted exclusively to lessons in appreciation of mathematics, just as the teachers of literature, music, and art give lessons in cultivating appreciation in their different fields. And all the time during the year as the work develops, I should be on the lookout for opportunities to introduce these illuminating, humanizing, and stimulating ideas into the regular lessons. Of course we won't get a one hundred per cent response, but neither does the teacher of literature, of music, or of art.

The recitation should be conducted according to the principles of scientific business management. Teaching is the teacher's job. Just as we expect a good employer of labor to surround his employees with proper physical conditions if he would get the best results, so the teacher should look carefully after the seating, lighting, temperature, and ventilation of the room in which his pupils are working. So far as possible, both in business and school, waste should be eliminated. The teacher should organize the routine work of the recitation such as the movements of the class, taking of attendance, passing of papers, etc., so

as to take as little time as possible. He should not waste the pupil's time through giving indefinite assignments, and through lack of adequate lesson planning. The head of a factory knows the output of his individual workers. The business man knows if there is one department of his business that is not paying dividends. He searches for the cause and changes his business methods. The teacher can use timed exercises and standardized tests to test the ability of the pupils, and to test the results of his own labor. He can discard a method that doesn't bring results and try another.

As teaching is the teacher's job so going to school is the pupil's job. The pupil doesn't come to school and then go out into life. The school years are life. He should be on time as the bank clerk, the salesman, the stenographer must be; he must bring his book, paper, and pencil to class as the carpenter brings his tools to his job, and the doctor his medicine case when he is called to see a patient. He should expect to prepare each lesson to the best of his ability, and to hand in his work the day assigned and not the next day. If a baker wishes to keep a customer, he makes good rolls every day and delivers the rolls for tonight's dinner today, not tomorrow. I have found pupils wonderfully susceptible to these parallelisms between their behavior as students and the practices of business people.

The underlying idea in the application of these business principles to teaching is most emphatically not the idea of mechanizing the recitation. If the many routine details of class room management can be easily disposed of, if right ideals and attitudes toward school work can be established, then and only then, can there be a minimum of friction during the class hour, the maximum time for, and freedom in, the handling of the subject matter of the lesson, and the greatest enjoyment, interest, and efficiency in the work.

The dominant aim governing class room procedure should be to keep every pupil thinking all the time. I wish to direct your attention to this thought through a series of pictures of recitations, and let you draw your own conclusions. In every case pupils were reciting on subject matter that had been assigned. A teacher was conducting a lesson in the solution by substitution, of systems of linear equations involving two unknowns.

One pupil after another was called upon to read off the solution of a problem from his paper. You recognize that there are four different first steps possible, so in all probability the solutions by all the members of the class could not have been alike. As each so-called explanation was given, the rest of the class sat in a passive attitude, only arousing themselves to compare answers.

In another class where the same type of lesson was under discussion, the pupils asked pointed questions about two or three troublesome problems. The teacher held the crayon and wrote the successive steps in the solution on the board, as she called upon various members of the class to think them out and state them. Every one seemed to be forming in his own mind the step he would give, and comparing it with the one that was given. No error was allowed to pass. Other possible solutions were suggested, and their values weighed. The results were checked. The general plan of solution was emphasized, and stated by one of those who didn't get the problem.

The lesson was in fractional linear equations. At the beginning of the hour the teacher called for answers; a pupil who had solved an equation and checked his result was allowed to give his answer. After all the results had been given, the teacher called the roll and each pupil responded with the number he had right. The teacher permitted those who had every problem right to work on the advance lesson, and sent the remainder to the board with the direction "attempt the first problem you had wrong." He passed quickly from one to another giving suggestions or asking questions. When several were perplexed about the same point, he called for the attention of the whole group at the board and explained the difficulty. He saw that he couldn't care for all those who needed help, so he called upon one of the stronger pupils to assist him. Is this an adaption of the method of instruction by homogeneous groups which any teacher may find it valuable to employ at times?

In a geometry class the different pupils were assigned propositions, and told to pass to the board and put on the figures. After giving eager, lingering, glances at the work in the book, they reluctantly set out on their tasks. As each one finished his drawings, he hastened back to his seat and studied. When

all had finished, the demonstrations began. Some pupils recited smoothly and correctly in a manner suggestive of memory work, and the others became hopelessly confused. The teacher assumed the responsibility for criticisms, and the class looked bored.

Let me present another picture of a geometry class. The teacher drew the figure for the proposition according to the pupil's direction. He always took such latitude as the direction permitted, and if the result was not the one the pupil desired, the teacher held him for the correct statement before making the teacher held him for the correct statement before making the drawing. Then the following questions were asked: (1) State the theorem. (2) What is given? (3) What is to be proved? (4) Give the general plan of solution. The proof was now given step by step by the different members of the class. Then one of the pupils was called upon to give the entire demonstration. He stepped quickly to the board where he could identify himself with the figure, and talked to the *class,* not to the *figure* or to the *teacher.* When he had finished, he remained at the board to answer questions, and defend his work against the attacks of his classmates. Everyone was attentive and interested. The teacher told me that frequently the pupil asked to recite, drew the figure free-hand himself, talking as he worked.

It was the week before the semester examinations. In one class room the pupils were feverishly going over the pages of propositions in geometry and problems in algebra. In another the review was a *new* view, for the teacher had given out sheets of questions which helped the student to get a grasp of the work as a whole. For instance, at the close of the year, the questions in ninth grade algebra centered around the equation.

Another teacher believed in reviewing units of subject matter as they were completed. After the class had finished long division, he asked them to make out a list of the definitions and laws and rules which they needed to know in order to work understandingly a problem in long division. In geometry when the class had studied the theorem, the sum of the angles of a triangle is equal to two right angles, the dependence of this theorem upon others, and of these upon still others, and the fact that every proof is ultimately grounded in definitions,

axioms, and postulates were admirably worked out as a class exercise in the making of a "geometrical tree." After the class had studied the theorem about the area of a circle, the teacher directed them to go back to the rectangle and trace the sequence of all the theorems involving areas, and carefully show the connections.

The ultimate aim of instruction should be to produce in our pupils scholarly attitudes of mind, and make self-reliant workers. How to attack the problems and plan their solution can profitably make an entire lesson exercise for many recitation hours in algebra, and how to attack geometrical theorems and exercises is the very foundation of all really efficient work in geometry. I have visited classes conducted on the syllabus plan where every theorem was an original exercise to be worked out analytically by the members of the class. In geometry these lessons on methods of attack are *essential* when the pupil is first introduced to the nature of a logical proof, and when some new type of proof, or some new subject, as loci or variables and limits, is taken up.

Mathematics furnishes us with a type of thinking which should be helpful in many lines of inquiry. Early in my teaching I remember how pleased I was when a geometry pupil of mine was asked by his English teacher where he got the idea for the form of an argument he had submitted, and he replied, "We always do that way in geometry." Without any direction from the teacher this bright lad had generalized his experience in geometry and made it available in debating. But the findings of psychology point to the conclusion that we should *teach* the pupils how to apply their reasoning in geometry to other fields; in general, unless related ideas are pointed out, they will not be perceived. If the pupil is to be able to use the identity, $(a+b)(a-b)=a^2-b^2$, in finding the product of 64 and 56, if the idea of functionality in algebra, and the methods of attacking geometrical exercises, are to transfer to his solution of the problem, "which college shall I attend?" he must be taught how to generalize the fact, idea, and method, and apply them to the new fields. The possibilities in training pupils in a technique of thinking through their study of geometry are strikingly brought out by Elsie Parker Johnson in the April, 1924, number

of THE MATHEMATICS TEACHER in an article entitled, "Teaching Pupils the Conscious Use of a Technique of Thinking."

Similarly, if the student after he leaves school is to be expected to apply his knowledge of mathematics to the solutions of practical mensuration problems, he must be taught so to apply it while he is still in school. How many gallons of water are in this cylindrical cistern? What is the height of the flag pole in the school yard? How would you proceed to lay off a tennis court or a football field? In each case have the pupils give a detailed description of a workable plan, then have them do the actual work.

I fear that such assignments as "Work the next ten problems," "Take the next two propositions" are far too frequently given. The teacher's preparation of the lesson should be made before the assignment, not afterwards. Then and then only, can he be prepared to make the assignment definite, suited to the capacity of the class, and encouraging. To make a task encouraging, the pupil should be led to the point where he really wants to get at the work and knows how to attack it. Here is a good place for fitting in some of the material that will lead the pupil to see that mathematics is worth while. At the beginning of the course in algebra, it may be best to devote time in the assignment to teaching the new concepts and the new language of algebra. In geometry, the assignment can properly be in the nature of the development or research lessons referred to above. Not all the difficulties should be removed, but the pupil should be put into a condition where he can help himself. A minimum assignment for everybody, and honor problems for those who can do them are often incentives to increased effort.

We must not forget that the ultimate aim of instruction is to produce self-reliant thinkers—real students who can study things out for themselves. So we must gradually throw them upon their own resources. Give them a hard piece of work to be done without any help, and the next day give a short test to see how many really have mastered the assignment unaided. The ability to get ideas correctly from a printed page is one of the most valuable attainments that a pupil can take away from the school room. No matter what he does after he leaves school, he will find this power a great asset.

Closely associated with the power to get ideas correctly, should be the power to express those ideas correctly. Clear thinking and clear expression are inseparably connected. What is the thought in the mind of the pupil who talks about "the value of an equation"? Some pupils make no difference in the reading of of these two expressions, $-a^4$ and $(-a)^4$, or in these four:

$$\frac{a+b}{c-d},\quad \frac{a+b}{c}-d,\quad a+\frac{b}{c}-d,\quad \text{and}\quad a+\frac{b}{c-d}.$$

Why do we hear "three and one thousand four hundred *and* sixteen *tens a* thousands," "sum of the triangles *are*," "divide a line *into* extreme and mean ratio," "intregal" for "integral," "frustrum" for "frustum," etc.? I wish that the words transpose," "cancel," and "clear of fractions," could be eliminated from our ninth grade text books. We don't need them, and they are convenient shields for the indifferent thinker to hide behind.

In speaking of the contribution that mathematics may make to good English, Dr. J. Rose Colby, head of the literature department of our school, says: "In mathematics we are dealing with concepts and relations that make in all probability our nearest approach to absolute truth. To mathematics, therefore, we are wont to look for the severest training in exact thinking. But exact thinking is practically impossible without exactness in the use of terms and logical precision in the ordering of our thoughts. And it is training in the exactness in the use of terms, this orderly and logical movement from thought to thought, in the interest of truth itself, that constitutes the peculiar service of mathematics to language training. The teacher of mathematics who fails to demand and secure these things, who fails therefore to develop in his pupils a conscience in the use of words, a scrupulous niceness in the basing of conclusions on premises, has missed the best part of his business."

Finally, may we not say that if we can make our pupils eager in seeking after truth, if we can cultivate in them a respect and love for truth, and can help to develop in them the ability and the desire to use truth for themselves and in the service of others, we shall have realized the whole range of aims of the teacher?

HOW TO FIGURE AVERAGES WITH THE TOP OF A SHOE BOX; OR THE USE OF AN ALIGNMENT CHART IN AVERAGING

By J. A. ROBERTS
The Wheeler School, North Stonington, Conn.

We have all admired the slide rule as a device for multiplying and dividing, squaring, cubing, and taking roots with a minimum expenditure of brain energy on our part. There is something attractive in manipulating a mechanical device and producing a result automatically. It is too bad that a slide rule will not add and divide so as to figure our averages for us; but add it certainly will not.

During my vacation the above truly idle thoughts led me to consider the value of the top of a shoe box, together with a few pins and a straight-edge, as an averagometer. My reflections led to a little interesting mathematics, even if the use of a shoe box top involves more work than the old way of figuring one's averages by arithmetical methods. With the idea of offering a bit of mathematical entertainment, rather than something of revolutionizing value, I submit the result of my leisure's cogitations.

Readers who are familiar with the idea of alignment charts will immediately see the source of my ideas. For those who have never heard of alignment charts I recommend Wentworth-Smith-Schlauch's "Commercial Algebra" (Ginn), or Mr. Ralph Beatley's course "The Teaching of Senior High School Mathematics" at a Harvard Summer School session. I make these recommendations in order that someone may read, digest, and possibly produce a better averagometer than mine.

The shoe box top must have a plain white surface. About an inch from one end draw a line across the width of the box. Divide this line in a dozen or so equal parts; the more numbers you intend to average, the more parts. Draw perpendiculars to the base line at these points of division, extending them to the limits of the shoe box top. Now, assuming that you are going to average grades, divide each perpendicular, equally, into twenty equal parts, and number these parts, in 5% steps,

from zero to 100%. Or, you may have a separate axis (perpendicular) on which the numbers may be put, as in the diagram (Fig. 1).

In order to explain the use of the device, let us average 60, 70, 50 and 90 with it. Provide yourself with one more pin than the number of terms that you wish to average; in this case, you

THE ALIGNMENT CHART FOR FIGURING AVERAGES

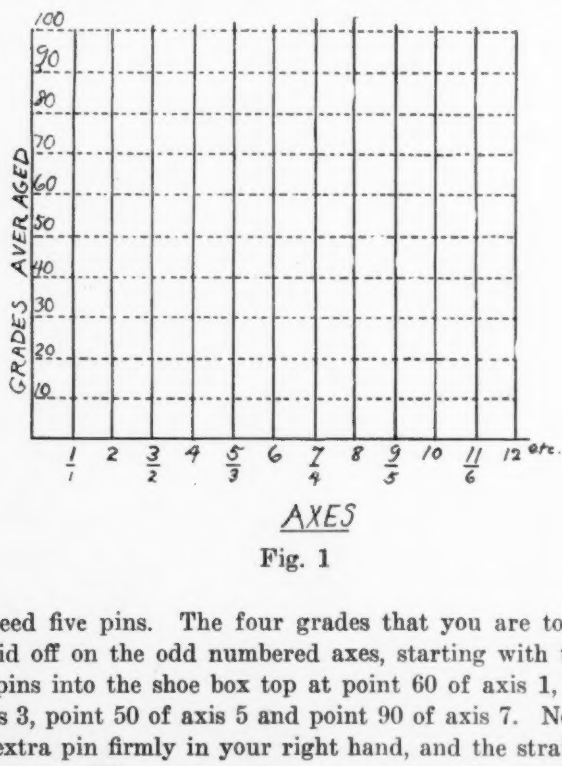


Fig. 1

will need five pins. The four grades that you are to average are laid off on the odd numbered axes, starting with the first; stick pins into the shoe box top at point 60 of axis 1, point 70 of axis 3, point 50 of axis 5 and point 90 of axis 7. Now grasp your extra pin firmly in your right hand, and the straight-edge in the left. We are now about to average the numbers.

Place the straight-edge against the pins occupying the first and third axes. On axis 2, note that the straight-edge indicates the average of two terms to be 65. Impale this 65 point with the extra pin, and remove the pin in axis 3. Disregarding the pin in axis 1, place the straight-edge against the average of 2

terms (the pin at 65 in axis 2) and the third term—50 in axis 5. Note that the straight-edge indicates, on axis 3, that the average of 3 terms is 60. Impale this 60 with your spare pin—the one removed from axis 3. Now remove the pin in axis 5. Place the straight-edge against the average of 3 terms (60 on axis 3), and the fourth term—the 90 pin in axis 7. On axis 4 the straight-edge indicates 67.5 as the average of 4 terms, and your result. It is obvious that, with axes enough, more terms can be averaged in similar fashion.

Now as to the mathematics involved in the idea. If we used the method of average any number of terms—say n terms—the result would come out on axis number n. The nth term averaged would have been laid off on axis number $2n - 1$.

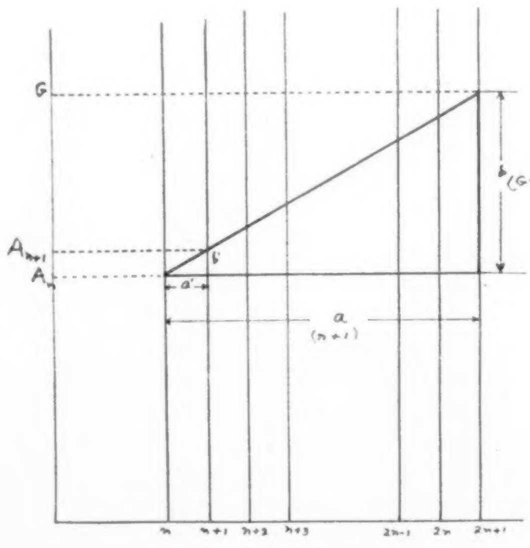


Fig. 2

If now we average in an $n + 1$ st term, it will be laid off on axis number $2n + 1$. (See Fig. 2.) In order to get a new answer, an average of $n + 1$ terms, we place a straight-edge against the average of n terms and the $n + 1$ st term on axis number $2n + 1$. The answer will be the intersection of the straight-edge with the $n + 1$ st axis.

If we draw a perpendicular from the average of n terms, on axis number n, to the $2n+1$ st axis, and consider the parts of axes $n+1$ and $2n+1$ intercepted by this line and the line of the straight-edge, we shall see that we have two similar right-angled triangles. Let us call the horizontal legs a and a', and the vertical legs b and b'. The length of b', added to the average of n terms (A_n) will be the average of $n+1$ terms. G is the $n+1$ st grade, averaged in last; the diagram represents it to be higher than A_n, but similar mathematical reasoning would apply were it lower; in the latter case we should get a length b' to subtract from A_n.

The length of a' is one unit of our horizontal axis. The ratio of a' to a is $1:(2n+1-n)$, or $1:n+1$. By plane geometry, this ratio equals the ratio of b' to b; b' is what we are seeking, and b equals $G-A_n$ when G is above A_n. When G is below b will be A_n-G; and when A_n and G are equal there are no similar triangles and no argument; for the average of $n+1$ terms A_{n+1} and of n terms, A_n, will obviously be the same.

$$\frac{1}{n+1}=\frac{b'}{G-A_n}$$

Whence $$b'=\frac{G-A_n}{n+1}$$

Since $$A_{n+1}=b'+A_n$$

$$A_{n+1}=\frac{G-A_n+A_n(n+1)}{n+1}$$

When parentheses are removed A_n's cancel, and we have

$$A_{n+1}=\frac{nA_n}{n+1}+\frac{G}{n+1}$$

which is obviously the average of $n+1$ terms in terms of the average of n terms and an $n+1$ st term.

If G is lower than A_n the proportion will be $1:n+1$ as $b':A_n-G$, whence b' comes out $A_n-G/n+1$. But since A_{n+1} now equals A_n-b', the same final result will be obtained.

DRAWING FOR TEACHERS OF SOLID GEOMETRY

By JOHN W. BRADSHAW
University of Michigan, Ann Arbor, Mich.

The College Entrance Examination Board in its Document No. 108, containing a definition of the requirements in geometry adopted by the board on April 21, 1923, has included an appendix on drawing which opens with the following paragraphs:

"An important aid to the visualization of space figures is the ability to draw these figures on paper. The pupil should be trained to make neat free-hand drawings, inserting whatever construction lines are needed and supplementing the representation of the space figure by independent drawings of plane sections, whenever such contribute to clearness. Ruler and compasses may often be used with advantage, but no technical knowledge of descriptive geometry or the niceties of mechanical drawing forms any part of the requirement. The use of ruler and compasses in the examination is permitted and to a moderate degree, desirable but it is not prescribed.

"The accompanying drawings indicate the sort of space figures which, beside those commonly met in the text-books, the candidate may need to represent on paper in connection with the problems which appear below."

These space figures include prisms, pyramids, and frustums, and the simplest intersections of cylinders, cones, and spheres, and the problems relate to dimensions, areas, volumes, diedral angles, and the form of various plane sections. It is the purpose of these articles to discuss in detail the sort of drawing that the board has described, especially for the benefit of the teacher who must prepare pupils to take the board's examinations, but it is believed that the subject will have a very wide interest, for geometrical drawings has been sadly neglected in our institutions that prepare teachers of mathematics, and little has been written about it in English aside from technical books on descriptive geometry and mechanical drawing.

It may be assumed that the teacher of geometry is interested in all pictures designed to aid the imagination in visualizing geometrical relationships. Roughly we may divide such pictures

into three classes, the free-hand sketch, the mechanical drawing, and the photograph. We are chiefly interested in the mechanical drawing, that is the drawing made with drawing instruments according to definite rules, but the others should be noticed in passing. All three have their uses and the teacher would do well to cultivate an appreciation of their excellencies; the ideal preparation for teaching geometry would include practical experience in producing each of these types, as well as in constructing models. But few teachers have had the benefit of such training and it is hoped that this discussion may contribute toward making up the deficiency.

Some may contend that sketching belongs in the realm of art and has no relation to mathematics, but when a rough, quick drawing with pencil or crayon is all that is needed to bring out a geometrical truth, an elaborate construction with ruler and compasses may be a useless waste of time. This must not be taken as a justification of careless or thoughtless work, for there are definite geometrical principles underlying every sort of representation of space figures, a disregard of which may lead to false conclusions and work positive injury to the geometrical imagination. If one follows the artist's method and sketches from actual models, he cannot go far astray; American textbooks, on the other hand, furnish numerous illustrations of faulty drawing arising from the attempts of mere draftsmen to sketch from imagination. The photograph, standing at the other extreme, must not be thought of as above criticism; many a text-book reproduction of a photograph of a model shows an exaggerated perspective, a distortion overlooked because of the abundance of detail. We shall confine our attention in what follows to the mechanical drawing, for a knowledge of the principles underlying their execution furnishes a basis for intelligent criticism of both sketches and photographs as well. We drop these then with the remarks that sketching from geometrical models is a highly useful exercise, and that photographs are worthy of critical study.

The process which we think of as underlying the making of mechanical drawings is called projection, and it is common to distinguish two types, central projection and parallel projection. Shadows, if cast upon a plane from a nearby source of

light small enough to be considered a point, are good illustrations of central projection. In essence the method of central projection implies that lines are drawn from a point called the center of projection to the various points of the object and produced, their intersections with the plane furnishing the pictures of those points. A similar projection is produced on the photographic plate by rays of light from the object, the opening in the shutter serving for a point. In parallel projection the lines, projecting rays we call them, instead of passing through a point have a fixed direction. Shadows cast by sunlight are illustrations of this type, for rays of sunlight may practically be considered as parallel. In central projection the object may be between the center and the picture as in the case of shadows, or the center may be between the object and the picture as in the photograph, or the picture-plane may be between the center and the object, as when we draw on a ground-glass plate a picture of what we are able to see of an object through it. Central projection approaches parallel projection when the center moves far away.

Each of these types of projection possesses advantages of its own. Central projection furnishes the more life-like pictures, because if the eye is placed where the center was, the picture will produce a retinal image of the same form as that produced by the object itself. It is a characteristic of central projections that the pictures of parallel lines are not parallel but converge to a point. This peculiarity that we call perspective is a familiar element of photographs or drawings of landscapes in which a road stretches away toward the horizon. If, however, there is occasion to make measurements on a figure and compare them with corresponding magnitudes of the object, as is done, for instance, when one wishes to reconstruct a space figure from a picture, parallel projection possesses the advantage that such comparisons may be carried out more directly. So the architect, when he wishes to give directions to the builder, uses parallel projection; when he wishes to show the owner what his completed house will look like, he makes a perspective sketch, that is, he employs central projection. Also the principles on which rest the making of parallel projections are easier to comprehend and apply.

The figures which the board offers as samples of the kind of work the pupil should be able to do belong to this class of figures known as parallel projections and for the present we shall turn our attention exclusively to their construction. Later we shall describe one method of making perspective pictures, for it is a knowledge of central projection that is necessary for the criticism of sketches and photographs mentioned above.

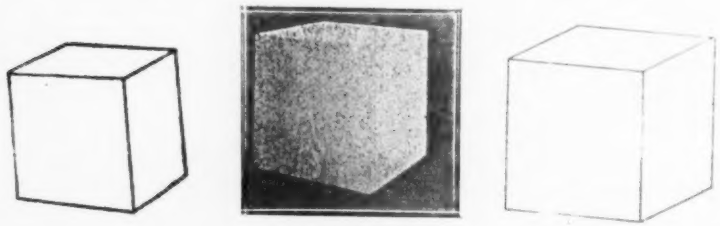


Fig. 1

By way of contrast we are placing side by side (Fig. 1) three pictures of a cube, a free-hand sketch in central projection, a mechanical drawing in parallel projection, and a photograph.

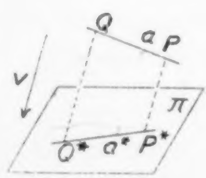


Fig. 2

If we are going to make a parallel projection of an object we must have a drawing board or picture-plane that we denote by π and we must decide on a direction of projection and this we indicate by an arrow v (Fig. 2). Then to find the projection of a point P of the object we pass through P a projecting ray parallel to v and find its intersection with π, a point that we denote by P^* and call the picture of P. If Q is another point of the object and the straight line PQ is an edge that we de-

note by a, it is clear that the projecting ray of Q and those of all the other points of PQ lie in the plane determined by a and the ray PP^*. This plane, the projecting plane of a, cuts π in a straight line a^*, the picture of a. Q^* and the pictures of all the points of PQ lie on this line. We have the important fact that the picture of a straight line is a straight line. There is one

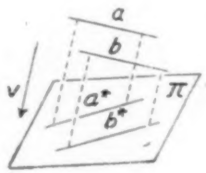


Fig. 3

exception. If PQ has the same direction as v, Q^* and the pictures of all the other points of PQ will fall upon P^*. In that case the projection of the line is a single point.

There are two characteristics of parallel projection that are a great help in making pictures. First, the pictures of parallel lines in space are themselves parallel. This is easy to see; for if a and b are two parallel lines of an object, each together with the direction v determines a plane, and these two planes, being parallel, cut π in parallel lines a^* and b^* (Fig. 3). An exceptional case arises when the projecting plane of a contains b; then the pictures a^* and b^* coincide.

Second, segments of the same line and their pictures are proportional, or to state it differently, if R divides PQ into two

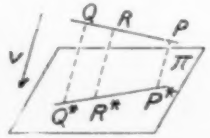


Fig. 4

segments PR and RQ (Fig. 4) that stand to each other in a certain ratio, then P^*R^* and R^*Q^* are in the same ratio. This is a simple application of the theorem about a system of parallels

cut by two transversals. A special case which is of frequent application is simply expressed in the statement, "The picture of a bisected segment is a bisected segment."

We may pause long enough to illustrate the usefulness of this second characteristic. Suppose that $P^*Q^*R^*$ is the picture of

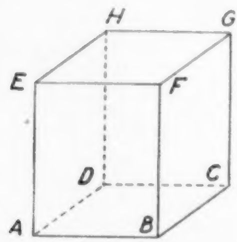


Fig. 5

a triangle PQR two of whose sides PQ and PR are known to be equal, and we wish the picture of the altitude PS. Since S is the midpoint of QR, S^* will be the midpoint of Q^*R^* and we can solve our problem by the direct construction on the picture.

It may be well to call attention specifically to the fact that perpendicular lines are not in general represented by perpendicular lines. It is clear, however, that any plane figure in a plane

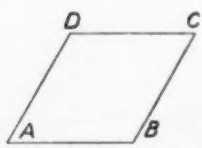


Fig. 6

parallel to π would have a picture that is congruent with itself.

Let us turn now to the first of the figures in the Boards Document (Fig. 5). Anyone seeing this figure alone would probably say, "That's the familiar picture of a square box or a rectangular parallelepiped." But the figure is accompanied by a description, it "represents a right parallelepiped the base of which is a rhombus" and in a supplementary figure is shown "the base with the correct proportions of the actual solid" (Fig. 6). The

picture is a correctly drawn parallel projection of the object described, but it is an equally good representation of a rectangular parallelepiped and it was natural to assign to the picture the most familiar object it could represent. The description and supplementary figure were necessary to convey the desired idea, though if the description had specified the angle at A, e. g. 60°, the auxiliary figure could have been dispensed with. The striking thing is that the picture does not suffice to tell us what the object was, or, as we say, we cannot reconstruct the space figure from one unaided plane figure. This is clearly a serious drawback if we wish to convey our thoughts by pictures. The landscape artist has so much of familiar detail in his picture that one can see what he saw, but the geometer who pictures bare outlines needs something more. It will be our next task to show how this deficiency can be supplied.

ANIMATED MATHEMATICS

By HOWARD BATES BAKER
The Lincoln High School, Jersey City, N. J.

Very much has been said about "intuitional mathematics." This, I believe, to be a misnomer, for it seems to me that there is very little intuitional about mathematics outside of the axioms and postulates. With due respect to Socrates, ask a boy to construct a square with an area equal to twice that of a given square, and watch the result.

The teaching dilemma in mathematics has been generally conceded to be the problem of so-called motivation (I abhor the parroting of these words). How to present to the student unequipped with scientific knowledge practical applications setting forth the need of the subject before entering upon the abstract principles, that is the question.

There are still those who say, "Away with your pampering. In our generation, we took it on faith, unsimplified, unabridged, and unadorned. A boy is an animal; and when his stomach is full, he is ready to play. The pursuit of knowledge can never be made anything but an obligation to him. You do not explain the physiological action of castor oil before administering the dose." And who can say that these men are not in the right?

However, if the student is to have a glimpse of the utility of mathematics, why not be less scrupulous about our applications? Why not tell the student some crude anecdote that will stimulate in his mind a vision of the possibilities in the principle under discussion? A great artist, I am told, does not paint the whiskers on the cow's chin.

In my teaching of mathematics, I have endeavored especially to invent some explanation of those features which I myself found perplexing as a student. One thing that bothered me was how there could be more than one correct answer to an equation. Even though they both checked, yet it seemed unreasonable. To meet this mental reservation of students, after I have worked out with them the solution of a quadratic equation by factoring, I tell them the following story. The same story I also tell to my geometry class just prior to the explanation of a locus.

A bandit, mounted on a bronco, was being hotly pursued by a

Sheriff's posse. He kept out of range for a long time; but, as the pursuit continued, he became conscious that the gait of his pony was beginning to waver. Now they skirted the edge of a little ravine, now swung up the steep slope of the embanking hill, clambering up the narrow trail cut through the brush; then, on a straight stretch, the pursuers gained; they opened fire. Swinging in his saddle, he dropped one of them; then another fell; but he himself swayed and nearly reeled from his saddle at the impact of a forty-four which tore a hole in his chest. In a few seconds he would be rounding a curve in the trail, and be obscured by the brush for a moment. With a grim

effort, he clung to his pony, and then, as they disappeared on the turn, he threw himself free of the horse, permitting him to speed on riderless.

Struggling to his knees, he dragged himself into the brush, agonized by his wound, and doubly so as it was scraped by the dry leaves laden with the dust of the plains. Spurred on by the desire to escape, though conscious of impending death, he crawled still farther till, to his astonishment and relief, when only a few yards from the trail, he found himself in a clearing and almost before the door of a trapper's cabin. With a last supreme effort, he dragged on and, with a thud, fell unconscious against the door.

The trapper, startled, opened the door, at first cautiously, and then, sensing the climax of a tragedy, hastened to lift the man to his cot and to wash and dress his wound, although his experienced eye did not fail to observe the hopelessness of recovery. Indeed, he had no idea that the wounded man would regain consciousness. But, even as at death the mind sometimes becomes

abnormally clear just before the soul starts on its long journey, so now the bandit stirred. He opened his eyes, dazed for a moment until he grasped the situation by reconstructing the thread of passing events. His gaze took in the bandage and other signs of care. Gaspingly he thanked the trapper: "Pardner, you are the only one who ever did me a good turn. They will soon track me here. Don't bother to bury me; let them do the dirty work. But before I pass out, I want to tell you where my gold is buried. About half a mile down the Brushtown Road, there is a big poplar tree. The gold is fifty feet from the tree and ten feet from the fence line." With that he gasped and was gone.

In the course of time, the trapper decided to find out whether the story was merely an illusion of approaching death. He fastened one end of a rope to a tree, and, measuring off fifty feet on it, swung the other end to a point ten feet from the fence line. There he dug, but he found no sign of gold.

As I describe the trapper's operations, I draw an illustration on the board similar to the one here given.

Shortly, the trapper's son, on vacation, returned from school in the city. He had studied algebra (or geometry, as the case may be). The boy was interested in the story, and explained to his father that there are other possible places where the gold might be within the bandit's instructions.

I now let the class lead me to the other possible locations, pointing out what conditions would make for two possibilities, what for three, and what for four. Usually some bright student will bring out that there is only one correct answer, as the gold is in only one place. The reply is that the problem is not, "Where is the gold?" but, "Where may the gold be to satisfy the conditions named by the bandit?" Thus it is clear that there may be more than one correct answer to a problem, and hence to an equation.

The interest is intense during the recital, and there is a psychological phenomenon. When I start to draw the figure, I ask, "Where is the gold located?" and every student has the figures on the tip of his tongue. Of course, a teacher with dramatic talent can intensify the recital.

As an illustration for geometry, of course, we make reference to the intersection of two loci.

Another example of approach by the story method is the teaching of the removal of a parenthesis.

In the removal of a parenthesis preceded by a minus sign, I follow the plan approved by so many successful teachers. The student is never told in any part of my work to change a sign. He is led to feel that, in order to remove the parenthesis, the indicated operation must be performed upon evéry term within the parenthesis. This mathematical feeling is developed by concrete illustration similar to that for the quadratic equation. The development is as follows:

Teacher—On his way home, a man spent $11 in one store and $6 in another, out of his weekly wage of $25. How much did he have left?

Student—Eight dollars.

Teacher—How did you find it?

Student—I added 11 and 6 and subtracted the sum from 25.

Teacher (writing on the board)—

$$25 - (11 + 6) = 25 - 17 = 8$$

That's good; but why do I use the parenthesis?

Student—To indicate that 11 and 6 must be added first.

Teacher—Is there any other way of finding the amount left?

Student—Yes. Subtract 11 from 25 and then 6 from 14.

Teacher (writing on board):

$$25 - 11 - 6 = 14 - 6 = 8$$

Which method do you prefer?

Student—The first way.

Teacher—Yes, the first method is the more customary; but how about this? (Teacher writes on the board)

$$25x - (11x + 6)$$

Student—We cannot add 11x and 6.

Teacher—That's right. It is impossible to combine 11x and 6, and we have no choice but to subtract each term in the parenthesis separately. (The teacher writes on the board)

$$25x - (11x + 6) = 25x - 11x - 6 = 14x - 6$$

Likewise,

$$10 + (7 - 5) = 10 + 2 = 12$$

and

$$10x + (7x - 5) = 10x + 7x - 5 = 17x - 5$$

It seems to me that a liberal introduction of these homely illustrations tends to animate the subject for the student.

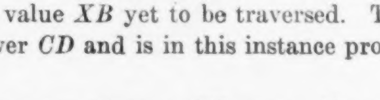
SOME METHODS IN SUBTRACTION

By CHARLES S. GIBSON
Principal, Theodore Roosevelt Junior High School, Syracuse, N. Y.

In teaching subtraction three factors of the subject present themselves to us in different ways, and the inter-relation of these factors by a series of permutations produces a variety of methods sometimes widely different, and sometimes very similar, but still enough unlike to prove confusing to children. First, there is the nature of the subject itself, second, the terminology to be employed, and third, the matter of "carrying" in the Arabic notation. The terminology depends in some cases directly upon the first factor and in others upon the third.

In looking at the nature of the subject let us discard for the moment the Arabic notation and the idea of unit value, and adopt the line graph as a convention to express quantities.

Now subtraction implies two given values, one, the larger, called the minuend, and a smaller, called the subtrahend. The problem is to determine the amount by which the one exceeds the other. Let these two values be represented by the lines AB and CD.



There are three methods of proceedure open to us.

One. Starting at A on the minuend we may move along to a point X on AB directly opposite D, or until $AX = CD$. There is *remaining* a value XB yet to be traversed. This value is the excess of AB over CD and is in this instance properly called the *remainder*.

Two. We may superimpose CD upon AB so that A *and* C coincide, then starting at B move toward A until we arrive at D (or X). The portion traversed is the *difference* between the two, and is the excess sought.

Three. We may extend CD to equal AB. The amount necessary to be added is likewise the amount of the excess of AB over CD, but in this case should be called the *increment*.

These three variations may be expressed also as follows:

One. Measure off on AB a portion equal to CD and cut it away. The part left is the *remainder*.

Two. Superimpose *CD* upon *AB* and measure the excess. It is the *difference.*

Three. Extend *CD* to equal *AB*. The amount added is the *increment.*

Suppose we assume three values, X, Y, and Z such that $X + Y = Z$. Now, if we have given Y and Z to find X, three solutions again appear:

One	$Z - Y = ?$
Two	$Z - ? = Y$
Three	$Y + ? = Z$

These three ideas may be given a concrete explanation.

One. John has ten marbles. If he gives William four how many will he have *remaining?*

Two. John has ten marbles and William has four. How many *more* marbles has John than William?

Three. John has ten marbles and William four. How many must be *given* William that he may have as many as John?

Suppose the teacher illustrates each of these three cases with actual boys and marbles. The first will require ten marbles, the second fourteen marbles and the third twenty marbles. The writer does not advise this be done. It would be confusing, perhaps, to children, but cites it to show the three ways in which the matter of subtraction may be viewed.

These three considerations give rise to our first set of terminologies.

One. (a) "Four from ten leaves six" or
(b) "Ten minus four equals six."

Two. (c) "The difference between ten and four is six."
(d) "The difference between four and ten is six."
In the first of these subtraction is *down* in the algorism, in the second it is *up.*

Three. (e) "Four and how many make ten?
Four and six are ten."
(f) "Ten equals four and six."

The Arabic notation with its decimal place value of figures gives rise to another difficulty in subtraction commonly called "carrying." When the system is once clear to the child the "carrying" in addition is easily understood, but in subtraction its explanation involves more difficulty and *five* plans have from time to time appeared for this purpose.

The first plan regards the *"carrying"* in subtraction as merely the *reverse* of that in addition; and in the explanation *subtraction* is considered to be simply addition reversed.

Suppose we have this example in addition. Add up.

64	The terminology used is
67	"Seven and four are eleven. *Write 1. Carry one.*
131	Seven and six are thirteen. *Write 13.*"

Suppose now we reverse the example giving only the sum 131 and the addend 61 to find the other addend.

??	The terminology of addition applied would be
67	"Seven and *what* are eleven? Seven and *four* (writing 4) are eleven. *Carry one.* Seven and what are thirteen? Seven and six (writing 6) are thirteen."
131	

This point may be made more obvious by a case in which the amount to "carry" is more than one. Suppose 241 is the sum of 63, 59, 24 and another addend to be found, or to state the example as subtraction suppose it is required to take the sum of 63, 59, and 24 from 241.

241	Terminology:	"4,13,16 and how many equal 21?
63		"16 and 5 (write 5) equal 21.
59		"Carry 2—4, 9, 15 and how many equal 24?
24		"15 and 9 (write 9) equal 24."

Because it is usual to have but one *subtrahend* and because the sum of any two digits never exceeds 18, it is equally unusual to have more than 1 to "carry."

This explanation of "carrying" results in increasing the digit in the subtrahend to the left of the subtracting place when the digit in the minuend is less than the digit in the subtrahend, and comes out in the final terminology exactly where the "equal addition" method brings us. If the "Austrian" or "Continental" method is to be confined to the idea that subtraction is the reverse of addition then this method and the terminology of addition characterize that method. Either of the other five terminologies may be used with this method of explaining "carrying."

A second method of explaining "carrying" is that of disintegrating the minuend.

Suppose we are to subtract 64 from 241.

$$\begin{array}{l} 241 = 200 + 40 + 1 = 100 + 130 + 11 \\ \underline{64 =} \quad \underline{\qquad\qquad\qquad\qquad 60 + 4} \end{array}$$

The usual terminology would be

241	"4 from 11 equals 7.
64	6 from 13 equals 7.
——	0 from 1 equals 1."
177	

As in the first method given either of the other terminologies may be used and a variety of methods result.

A third method depends upon the axiom that if equal numbers be added to both minuend and subtrahend the remainder will not be changed. This is called the qual addition method and is similar to the old time "borrow and pay back" method.

$$\begin{array}{l} 241 = 200 + 40 + 1 \\ \underline{76 = \qquad\quad 70 + 6} \end{array}$$

adding $100 + 10$ to both we have

$$\begin{array}{l} 240 + 110 = 200 + 140 + 11 \\ 76 + 110 = \underline{100 + \ \ 80 + \ \ 6} \end{array}$$

the terminology becomes

241	"6 from 11 leaves 5. One to carry.
76	8 from 14 leaves 6. One to carry.
——	1 from 2 leaves 1."

All the other terminologies may be employed in connection with this explanation, but the final result in the mind and practice of the child will be about the same as if the first method given above were used.

A method of solving examples in subtraction appeared in the "School Masters Assistant," by Thomas Dilworthy, published in England toward the close of the eighteenth century and in Mr. Pike's arithmetic published in America about the same time. Mr. Adams, in his arithmetic published in 1811, refers to the method as Mr. Pike's suggestion. It was revived and exploited in New York City about 1884 by Mr. Goddard. The following illustrations will be sufficient without giving the detail variations:

13	If a child had 13 cents, a dime and 3 cents and wished to spend
6	6, he would give the merchant the dime, receive 4 cents in change
—	and add it to the 3 he had to make a remainder of 7.

This method may be called that of *decimal complements.*

241 64 ——	"The decimal complement of 4 is 6. 6 and 1 are 7." Using the equal addition explanation of carrying we would say: "The complement of 7 is 3; 3 and 4 are 7; 1 from 2 is 1."
390412 276271 ——	"1 from 2 = 1 3 (comp. of 7) + 1 = 4 3 from 4 = 1 4 (comp. of 6) + 0 = 6 8 from 9 = 1 2 from 3 = 1

Another method somewhat similar in that it involves use of the decimal complements has been used in some parts of Europe. In this the smaller integer is always taken from the larger. If the smaller integer is in the minuend, the complement is set in the answer.

241	1 from 4 = 3	comp. = 7	
64	4 from 7 = 3	comp. = 7	ans. = 77
3096215	2 from 5 = 3		
2841392	1 from 9 = 8	comp. 2	
——	2 from 4 = 2	comp. 8	
254823	2 from 6 = 4		
	4 from 9 = 5		
	0 from 8 = 2	comp. 2	
	2 from 3 = 1"		

With three mathematical variations of the fundamental notions of subtraction, six distinct terminologies, and five plans of handling a situation with a smaller integer in the minuend all of which may be more or less intermingled, it is evident that a great variety of methods is possible, especially when teachers and text book writers individually select some special feature of the case to emphasize.

The following example may be explained in either of several ways:

$$\begin{array}{r} 41 \\ \underline{27} \\ 14 \end{array}$$

1. "Seven from eleven leaves (equals, equal, leave, are) four." Carry one. (One to carry); "three from four leaves one" ("carrying" being explained as inverse of "carrying" in addition).

2. "Seven from one I cannot take so I must add ten to one making it eleven. Seven from eleven leaves four. Since I've added ten to the minuend I must add the same amount to the subtrahend. This I do by adding one to the two making it three. Three from four leaves one."

This is the equal addition method.

3. "I cannot take seven from one, so I take one from the four and add its value, ten to the one making it eleven. Seven from eleven leaves four. Two from three leaves one."

This is the disintegration method.

4. "The complement of seven is three. Three and one are four. Since I used a complement I must carry one. Three from four is one."

5. "One from seven is six. The complement of six is four. Since I subtracted *down* I must carry one. Three from four is one."

6. "Eleven minus seven equals four. Four minus three equals one." (Similar to No. 1.)

7. "Seven cannot be taken from one so I must add ten to the one, making it eleven. Eleven minus seven equals four," etc. (Similar to No. 2.)

8. "Eleven minus seven equals four. Three minus two equals one." (Similar to No. 3.)

9. Repeat No. 4 except in the last statement. Say "Four minus three equals one."

10. Repeat No. 5 except the last statement. Say, "Four minus three equals one."

11. Regard "carrying" as the inverse of "carrying" in addition. Say, "The difference between seven and eleven is four. Carry one. (One to carry.) The difference between three and four is one."

12. Same as No. 11 except reverse order of digits. Say, "The difference between eleven and seven is four," etc.

13. Explain "carrying" as in No. 2. Use terminology of No. 11.

14. Explain carrying as in No. 2. Use terminology of No. 12.

15. Explain carrying as in No. 3. Say, "The difference between seven and eleven is four. The difference between two and three is one."

16. Same as No. 15 with order of digits reversed. Say, "The difference between eleven and seven is four. The difference between three and two is one."

17. Same as No. 4 except last sentence. Say, "The difference between three and four is one."

18. Same as No. 17 except say, "The difference between *four* and three is one."

19. Same as No. 5. Except say, "The difference between seven and one is six." "The difference between three and four is one."

20. Same as No. 5 except say, "The difference between one and seven is six." "The difference between four and three is one."

21. Same as No. 1 in explanation of carrying. Say, "Seven and how many are eleven? Seven and four (write 4) are eleven. Carry one. Three and how many are four? Three and one (write 1) are four."

This is the "Austrian" or "Continental" Method.

22. Explain carrying as in No. 2. Use terminology of No. 21. *This method is very close to the "Austrian" and some writers give it as such.*

23. Disintegrate the minuend and use the terminology of addition. Say, "Seven and how many are eleven? Seven and four (write 4) are eleven. Two and how many are three? Two and one (write 1) are three."

24. In No. 4 substitute for the last statement in the terminology: "Three and how many are four. Three and one (write 1) are four."

25. Say, "One and how many are seven? One and six are seven. The complement of six is four (write 4). I subtracted *down*. Carry one. Three and how many are four? Three and one (write 1) are four."

26. Explain "carrying" as in No. 1. Say, "Eleven equals seven and four (write 4). Carry one. Four equals three and one (write 1)." *Some authors offer this as the "Austrian" method.*

27. Explain carrying as in No. 2. Use terminology of No. 26. *This variation may easily be called "Austrian."*

28. Say, "Eleven equals seven and four (write 4). Three equals two and one (write 1)." *This is another variation which might be called "Austrian."*

29. In No. 4 for the last sentence substitute "Four equals three and one."

30. In No. 5 for the last sentence substitute "Four equals three and one."

A close examination will show the possibility of other variations.

In practice, the explanations of "carrying" given in Nos. 2 and 3 are soon dropped and methods offered as variations on that account become in practice, identical with them.

The *equal addition* method, No. 2, is very uniformly treated in text books which offer it as is the *disintegration* method, No. 3; but there is no such uniformity in the presentation of the "Austrian method." (See Nos. 21, 22, 23, 26, 27, 28.) This may account for the failure of this method to make a better general record in our schools.

Supporters of No. 2 claim there is no disadvantage in learning the subtraction combinations as a separate table, that the method persists in use, and that it coincides most nearly with the predominating problems in subtraction, that of finding a *remainder*. They point out that No. 3 is difficult whenever 00 appears in the minuend.

Supporters of No. 3 claim that method most easily comprehended by children and point out that its use is universal in mixed numbers in case the fraction in the minuend is smaller than that in the subtrahend.

Supporters of Nos. 2 and 3 agree in opposing the Austrian as equally difficult with either of these two in that it is about as much of a task for a child to learn the addition combination *backward* as it would be to learn a new set. They point out that the terminology (see No. 21) is clumsy. If the question is omitted, as is often the case, the language is not in the *order* of the mental process which required the answer to come at the *end* of the statement. Children taught this method are apt to set the *minuend* figure in the answer, especially in long division. They point out also, that the "Austrian" method has not been justified in practice.

Supporters of the "Austrian" method claim mental economy because of the use of the addition combinations already learned contending that addition and subtraction should be taught together. They usually neglect entirely to give the required explanation of carrying, and have weakened their case by allowing so many variations to come into use.

That a uniform method, especially in the same school system and certainly in the same school is desirable goes without question. As a pupil goes from teacher to teacher such a process should be handled in a common fashion. But this is not always the case, and the result is confusion. This is especially true when the child is first introduced to the subtraction of mixed numbers. Here, whatever method he may use before, he is usually taught to *disintegrate* the minuend to obtain a fraction larger than the one in the subtrahend.

$$16\tfrac{1}{4} = 15\ 5/4 \qquad\qquad 10\tfrac{1}{8} = 9\ 9/8$$

or

$$8\tfrac{1}{2} = \ 8\ 2/4 \qquad\qquad 5\tfrac{1}{2} = 5\ 4/8$$

Five methods of handling this case follow. These are all subject to variations in terminology.

(1)	16 1/2	Say "1/2 and ? = 1 1/4 (5/4)
	8 1/2	1/2 and 3/4 = 1 1/4 (write 3/4)
		9 and ? = 16
		9 and 7 = 16 (write 7)"

The reduction is mental; the carrying is that of addition reversed; the language that of addition. It is the "Austrian" method applied to mixed numbers.

(2)	16 1/4	16 1/4 + 4/4 = 16 5/4	Say "2/4 from 5/4 = 3/4
	8 1/2	8 2/4 + 1 = 9 2/4	9 from 16 = 7"

This is the *equal addition* method.

(3)	16 1/4 = 15 5/4	Say "2/4 from 5/4 = 34
	8 1/2 = 8 2/4	8 from 15 = 7."

This is the *disintegration* method.

(4)	16 1/4	Say "1/2 from 1 (a unit) = 1/2
	8 1/2	1/2 + 1/4 = 3/4
		9 from 16 = 7."

This is method No. 4 above extended to mixed numbers. Notice that the *unit*-complement of the fraction in the subtrahend is used.

Another illustration of this same method might make it clearer.

	10 1/8	Say "1/4 from 1 = 3/4
	5 1/4	3/4 + 1/8 = 7/8
		6 from 10 = 4."
(5)	16 1/4	Say "1/4 from 1/2 = 1/2
	8 1/2	1/2 from 1 = 1/2
		9 from 16 = 7."

This is method No. 5 applied to mixed numbers. Another illustration will perhaps make it clearer.

10 1/8	Say "1/8 from 1/4 = 1/8	
5 1/4	1/8 from 1 = 7/8	(write 7/8)
	6 from 10 = 4	(write 4)"

Subtraction of denominate numbers is usually done by disintegration, but any of the methods given here may be applied. The subject is not of sufficient importance to warrant discussion.

CHANGES IN SUBJECT MATTER AND METHOD TO FIT DIFFERENT ABILITY GROUPS IN MATHEMATICS[1]

By LILLIS PRICE
New Trier Township High School, Kenilworth, Ill.

In presenting the changes in subject matter and method needed to fit different ability groups in high school mathematics, I am compelled to limit my discussion to the work of the first two years as carried on in the majority of schools: namely, freshman algebra and plane geometry. My experience with ability groups limits me to this field and since a large number of high schools make two years of mathematics required for graduation, it is the field where different ability groups are most needed and the field in which I feel you would be most interested.

It is also my intention to deal only with the below-average group, which we call the slow-moving group, and the above-average group, which we call the accelerated group. Our subject matter and methods have always been more or less adapted to the normal or average group—and it is these others that are claiming our attention at present.

In choosing either method or subject matter adapted to any group, we must *first* study our group, find out its outstanding difficulties and abilities, then make our instruction meet them.

In the slow-moving group, we find a group of boys and girls who have, usually, met failure, found out the discouragement and criticism which accompanies it and are, consequently, a little afraid of the new job to be tackled. To break down this fear and replace it with a confidence that they are capable of doing the job set before them, is the first duty in connection with the choosing of subject matter and method of instruction.

In algebra, the usual subject matter at the beginning of the course is simple enough to produce confidence—when coupled with a patient and sympathetic instructor who is willing to go *slowly* and explain and explain again and again. This necessity of *repeated explanation* is an outstanding feature of the method of instruction for slow-movers.

In geometry, we are successfully breaking down the fear approach, by a series of simple problems in the handling of geomet-

[1] Read before the Principals and Superintendents Section of the High School Conference at the University of Chicago, May 9, 1924.

ric tools and the discovering of geometric relationships. For instance, the first one is: "Draw several triangles and measure the angles. Find the sum of the angles of each triangle." This simple exercise gives us, of course, a basis for the introduction of geometric tools and a great many geometric concepts. The experience of working with them coming first and out of that experience arising the need for the statement of geometric truths and later their proof. The work for the first six or seven weeks is done without the use of a textbook, merely, through carefully graded and selected material which first gives the pupil a geometric experience and then helps him to formulate this experience in geometric terms, and finally leads him to prove that, under the same conditions, the same results will always be obtained. At the end of the six or seven weeks, the boys and girls are well acquainted with the idea of an angle, triangles of all sorts, perpendiculars, parallels and parallelograms, etc., and can construct them much more readily than a normal class at the end of the same period. They have had a real experience in connection with them and consequently a real understanding of their meaning. In addition to this they have acquired the ability to prove simpler types of exercises and theorems, where a real necessity of proof was apparent to them. This manner of approach has resulted in a liking of rather than the fear of geometry, with which the slow-mover entered the class.

The second great problem in connection with the teaching of a slow-moving class, is the problem of keeping each *individual working*. An inability or else an unwillingness to think with the group or with the instructor seems to be an outstanding characteristic of the below-normal group. Perhaps this is a result of having too much instruction "over their heads"—or perhaps it is merely a tendency to loaf—at any rate, it represents a real problem to the teacher of the slow-moving group, and whatever method of instruction she uses, it must meet this problem. To meet this difficulty, Mr. Stokes, of New Trier, is trying the "individual method" of the Winnetka type with a slow-moving algebra class. He has not completed one semester's trial and we are not, therefore, ready to praise or condemn the experiment. Others have tried combinations of the individual and class instruction. This combination of individual and class in-

struction has been found quite helpful in geometry instruction. By taking a few minutes at the beginning of the period to correlate the previous day's work with the preceding day's and to analyze the problems, which the instructor has carefully selected for the day's work, the class is then set to *work* for the period, taking out as home-work whatever they fail to complete during the period. In this way, each individual in the class is compelled to work and think independently as well as with the group.

The third difficulty which the subject-matter and method of instruction should surmount is, of course, the inability of this group to master subject matter of too complicated a character. Subject-matter must, therefore, be limited to the bare essentials and problems must be chosen which do not involve too many different principles. If one type of work is taken up and explained and explained again and again, even the dullest are capable of mastering it, provided they are not given too many different things to remember at one time.

The rudiments of algebra which this group of boys and girls will need can be taught in one year, provided it is not adorned with a great many complicated problems—of little practical value and serving no real purpose for this particular group. The subject-matter taught should cover: addition, subtraction, multiplication and division of signed, literal and fractional numbers; solution of simple equations in one and two unknowns—including the use of a formula and its transformation; square root and simplification of radicals, and solution of quadratic equations in one unknown by factoring and completing the square.

In geometry, the subject-matter should consist of the theorems and exercises where the proof is of the direct type. The proofs of theorems where the necessity of proof is not easily evident to the slow-mover and where the proof requires finer thinking than they are capable of, should be omitted. Emphasis should be placed upon original proofs of theorems and exercises and the pupil should be taught to work out his proofs by analysis.

The chief pleasure in teaching such a group lies in the fact that, having once mastered material, they do not forget it as rapidly, perhaps, as those who master the work more readily. In one of my geometry classes, the other day, we came to some

problems where it was necessary to take the square root of a number. One of my algebra slow-movers from the previous year, was the first one in the class to work the problem.

In the accelerated groups, we find entirely different types of difficulties and abilities. This group can think with the rest of the group and think independently. They are especially adept at expressing themselves in mathematical language and show almost uncanny ability in thinking ahead of the work at hand. It is great fun to let them take the lead in selecting material to be studied. A few days ago we were reviewing raising of monomials to different powers. Then the suggestion was made—well what about binomials. We easily obtained the square—then cube and then fourth power. Well, first thing we knew we were plunged into the binomial theorem and then we had to $(a+b)^n$th power and finally, I asked how we could obtain one term of the expansion without all the preceding terms—say the fifth. One of the boys promptly produced it and then a little girl, with dresses above her knees—raised her hand and proceeded to state the method of finding any term of the expansion $(a+b)^n$.

All of this, of course, makes the class most interesting to students and teacher alike, but—two big difficulties must be kept in mind in the instruction and choice of subject-matter—(1) that which comes easy is apt to go easy. These youngsters need a sufficient amount of material—more than the average student—to really *acquire* the information they frequently get by inspiration. During the first two years of ability grouping at New Trier, the accelerated classes were given the regular year's work of freshman algebra plus the half year of advanced algebra, usually taught in the junior year. They literally ate it up, passed better exams, some of them, than the juniors. However, some of them are now in the college algebra and college review classes, and we find they do not measure up to the students who have taken the normal math courses. They think they know it, but they don't, which brings me up to my second difficulty with the accelerated group—(2) the tendency to over-estimate their own ability, which is apt to hinder their progress later.

To meet these difficulties, we are not giving the accelerated class this year, the junior algebra. We are covering the topics

of freshman algebra, but are covering them with great thoroughness and delving into all the possible cases in connection with them. We are covering practically every problem in the text and more besides. I find that the class particularly enjoys contests—so we have contests whenever we can find the slightest pretext for them. In other words we take plenty of drill work in the most agreeable doses we can find. The class recitation period becomes one of two kinds—either a discussion period in connection with the introduction of new material, or the explanation of old material by some pupil, or a drill period, disguised as contests of various sorts. A great deal of homework is assigned and these are closely checked up on. This particular type of pupil needs to learn to *work* and work hard.

The subject of geometry offers such a wide field of supplementary material in the way of exercises and problems that the accelerated group find plenty of subject matter to work upon. Here drill work is not so necessary, and ingenuity and independent thinking have the greatest sway.

Difference in ability within, even the slow-moving group, and accelerated group is very evident and just as evident is the difference in ability between different groups of slow-movers and between different groups of accelerated pupils. Each teacher must study her own group and be ready to use the subject matter and method it seems to demand. I have only attempted to pick out the outstanding difficulties of all the ability groups I have handled, rather than the particular ones of each different group.

In conclusion, let me add, that, after three years of experimenting with ability grouping at New Trier, we are convinced that the instruction given to a group of carefully selected slow-movers is much more effective and produces much better results than the same amount of time spent in instruction in mixed classes upon the same group of individuals. We are not so enthusiastic in our recommendation of the accelerated groups. In all probability we shall not have a group of accelerated people next year. We feel that our normal groups, especially in geometry, will be benefited by their presence in the class and that geometry offers a sufficient wealth of material to keep them busy and interested in a normal group.

NEW BOOKS

The Pilot Arithmetics. Book One and a Teacher's Manual for it by LOU BELLE STEVENS and JAMES H. VAN SICKLE, and Books Two and Three by HARRY B. MARSH and JAMES H. VAN SICKLE. Newson & Company, New York and Chicago.

These books possess several attractive features. They contain very little superfluous material; the proportion of problems to exercises and practice pages is well-balanced; the gradation of difficulties is pedagogical; the recurrence of number facts is psychological; and the approach to each subject is direct but clear, with no unnecessary explanations and no development too difficult, technical, and tedious for the pupils, except the method of subtraction. It is strange that the non-teachable method of subtraction (the method of equal additions) is chosen by some authors instead of the teachable "borrowing" method (the method of decomposition). What is the advantage of the former method over the latter in accuracy and speed? Which is the more intelligible method for both the teachers and pupils? Should pupils operate blindly with numbers, even when the numbers are small enough to make the procedure intelligible?

The Pilot Arithmetics are less padded with unsuccessful attempts to play the role of the teacher than many of the recent texts. The teacher should lead the way into arithmetical subjects in the grades, not the text. The text should aid the teacher by supplying and arranging problems, exercises, and practice material, not attempt to displace her. Some recent texts attempt to develop each topic by means of a series of questions, directions, and wordy suggestions which are easy to understand if one already understands the topic, but too difficult, too tedious, and too technical for the pupils to follow. We will learn some day that a large part of arithmetic is quite technical and cannot be written so that it can be self-taught to a large majority of the pupils. No text in arithmetic can take the place of a good teacher. The teacher should develop each topic without the use of a text and then use the text as a source of practical problems and practice material carefully selected and pedagogically arranged. Authors of so-called self-teaching arithmetics should refrain from trying to write texts to take the place of the teacher

by including detailed methods of presentation and even the teacher's questions arranged in developmental order. The pages and pages of book-method should be omitted, for the pupils will not read them and if they do they usually do not understand them and must be taught by the teacher. If the authors wish to give teachers suggestions as to method, let this be done in a teacher's manual, not in a text for the pupils. The authors of Book One of the Pilot Arithmetics have prepared a teacher's manual to go with this book. It is filled with helpful suggestions on methods and material for the first four grades.

E. C. HINKLE.

The Brown-Eldredge Arithmetics. By JOSEPH C. BROWN and ALBERT C. ELDREDGE. Row, Peterson and Company. Chicago and New York.

The three books of this three-book series are "practical, usable textbooks, free from fads"; they contain a "maximum of practice and a minimum of theory." The development of each topic is simple and natural, with all technical explanations omitted.

It is interesting to see that the authors of these texts recognize the true function of a textbook in arithmetic. Text books in arithmetic that attempt to take the place of the teacher by actually asking the teacher's questions in the development of a topic do not belong in the hands of pupils. Such are teachers' manuals on method and should be used as such. A good textbook in arithmetic should present a pedagogical sequence of the different topics and a satisfactory gradation of the difficulties of each topic together with a sufficient number of practical problems and ample practice material arranged in accordance with the latest conclusions resulting from investigations. This set of text-books recognizes the functions of the teacher, usurps none of her duties, and provides an abundance of problems and practice material for every topic. These texts contain more practical problems for every topic than are usually found in similar texts.

Book Three contains material suitable for the seventh and eighth grades of the regular elementary school or of the junior high school. The following junior high school topics for seventh and eighth grades are given sufficient treatment and attention to satisfy all except extremists; the formula, the equation, the

graph, the metric system, indirect measurement of line-segments by means of similar figures (a little scale drawing in the middle grades), the measurement of angles with the protractor, and a minimum of intuitive geometry in connection with mensuration (but no constructional geometry).

These texts are so satisfactory in every way that it may appear trivial to object to the perpetuation of such topics as the Roman notation, the six percent method of computing simple interest, and partial payments. In spite of all the possible objections, trivial or otherwise, against parts of these texts or the general plan, they are exactly what teachers of arithmetic are clamoring for.

E. C. HINKLE.

Junior High School Mathematics: Revised Edition. By VOSBURG, GENTLEMAN and HASSLER. The Macmillan Company, 1924.

Seven years ago Messrs. Vosburg and Gentleman published a three book series of texts in mathematics for the junior high school. These texts were unique in the success with which they correlated or combined into what is now called *general* mathematics, the essential elements of arithmetic, intuitive geometry, algebra and trigonometry. After seven years of use in the classroom, they have now been revised. Professor J. O. Hassler of the University of Oklahoma has succeed the late Mr. Gentleman as the co-author with Professor Vosburg.

The revised *First Course,* for the seventh school year, gives about equal emphasis to arithmetic and to intuitive geometry. The arithmetical work contains adequate well graded, timed exercises and many lists of oral exercises; it emphasizes checking, estimating of results, and every day applications of percentage. The intuitive treatment of geometry is well organized. Sufficient algebra is taken up to enable the pupils to express the relationships of intuitive geometry algebrically. One topic of great importance in the development of mathematical ability is *ratio.* This topic receives more than ordinary emphasis, and except for the treatment on page 78 it is commendable.

Graphic representation is introduced early in connection with the representation of progress on timed tests, and is stressed throughout.

The revised *Second Course* continues the work of the first book. The arithmetical work is organized around the home, farm, city and other activities; the geometry is more formal and more systematic; the simple equation and the formula receive much attention.

Two topics that are usually included in the second book of junior high school mathematics texts have been excluded, negative numbers and trigonometric ratios. These books emphasize approximate computation, or common sense in computation, more than any other text known to the writer.

In listing text books in junior high schools, teachers will doubtless give the Vosburg-Gentleman-Hassler series a high rating.

NEWS NOTES

Professor David Leib of the Connecticut College for Women addressed a mathematics section of the Connecticut State Teachers' Association, at New London, October 24th, on the topic, "Pitfalls in Algebra." Professor Leib pointed out that while high school teachers were forced to produce, from a heterodox group, a group of students of intelligence, with which the college teacher could go on, that there were defects, in students and in the teaching, which from the standpoint of the college teacher ought to be remedied.

Professor Leib's contention was that we are not coupling up common sense with algebra. All that is necessary to learn mathematics is common sense plus a few new words. An examination of college freshmen, even those coming to college with high secondary scholastic records, revealed startling discrepancies in common sense possession in regard to mathematics topics. For instance only 50 per cent of the students answered with correctness the following questions:

Mark true or false (topic: radicals):

$(a^2)^{\frac{2}{3}}$ equals $a^{\frac{8}{3}}$

$$\sqrt[3]{2} \cdot \sqrt[6]{2} = \sqrt[9]{4}$$

$$(a+b)^0 = a^0 + b^0 = 1 + 1 = 2$$

$$a^{\frac{2}{3}} \div a^{\frac{1}{2}} = a^{\frac{1}{3}}$$

$$\frac{a(a+b)+2}{a+b} = a+2$$

The student would seem to be able to do the work if he could have some one to tell him what "case" his problem came under or if he could remember the time when problems like the presented one were done. Initiative is seemingly lacking.

The lecturer gave five pitfalls of which an intelligent teacher should beware. 1. The humanizing pitfall. Under the supposition that we are humanizing mathematics we are giving the student many trivial applications in a field or in fields of which he is ignorant. 2. The enriching pitfall. Under the supposition

that we are broadening the student we spread out our courses over a wide range, instead of making for enrichment by deep digging over a short area. 3. The pitfall of wrong objectives. We should not eliminate the hard work to make the class average come up, or let our courses be steered by the C. E. E. B. examinations, for instance. 4. The pitfall of too many rules. A rule is easy to teach, but it is the hard taught principle that is of value. 5. The one-way pitfall. We should stress *clarity of thought* without regard for a fixed method or predetermined stereotyped program of course procedure. (Contributed by J. A. Roberts.)

"Why the Disciplinary and Cultural Values of the Modern Languages Should Be Stressed" appears in *Hispania* for March, 1924. This is a summary of an address given at the annual meeting of the American Association of Teachers of Spanish, in New York City, December 31, 1923, by Professor E. C. Hills, of the University of California. Quoting:

"Recent experiments, which have been made with greater skill and with more precision than those which were made ten or fifteen years ago, have shown that there really is such a thing as general training of the mind as the result of disciplinary studies. This general training is manifest in part in the development of patience, power of concentration, and habits of thoroughness. And there is transfer of power—not the complete transfer that was claimed fifty years ago, but a modified transfer to related fields.

"In view of these findings the more advanced thinkers in education are again advocating disciplinary studies in the public schools—such studies, for instance, as mathematics, foreign languages, and the natural sciences. Unfortunately, a very considerable proportion of our public school men seem still to be unaware of the results of the latest investigations in educational psychology; they oppose disciplinary studies and favor those that are basically informational. This is, I believe, one of the chief causes of the mental flabbiness of our boys and girls who receive their training in the public schools."—(A. D.)

FOUR articles of interest to high school teachers of mathematics are:

"Why Educational Objectives? *School and Society,* for May, 10, 1924. An address given before the National Society of College Teachers of Education, by Professor B. H. Bode, at the Chicago meeting, February 25, 1924.

"The School of Education and the University," *School and Society,* for July 26, 1924. An address by Professor Florian Cajori at the dedication of Haviland Hall, University of California, March 25, 1924.

"College Scholarship and the Size of High School," *School and Society,* for August 9, 1924. An investigation by Lester H. Thornberg, Samoa. Summary of a Master of Arts thesis.

"Latin Coming to Its Own," E. E. Cates of Los Angeles, California. Concluding paragraph: "Secretary of State Charles E. Hughes frankly declared before the N. E. A. at Boston that he did not think a satisfactory substitute had been found for the classical and mathematical training."—(A. D.)

The Association of Mathematics Teachers of New Jersey held their twentieth regular meeting at the Battin High School, Elizabeth, N. J., on May 10th, 1924. The program included:

1. A Review of Some Mathematical Researches Reported at the Chicago Convention, by Principal Robert Speer, Nishuane Junior High School, Montclair.
2. Typical Troubles in Junior High School Mathematics, by Miss Hattie Smith, Junior High School, New Brunswick.
3. Presidential Address: The Teaching of Verbal Problems, by Mr. Howard F. Hart, High School, Montclair.
4. Curves with n Real Asymptotes and an $(n—1)$ Multiple Points, by Professor Richard Morris, Rutgers College.
5. Theorems Relating to Simson's Line, by Mr. Louis M. Paradiso.

The officers of the Association are: Howard F. Hart, President; Professor Oswald Veblen, Vice-President, Princeton University, and Andrew S. Hegeman, Secretary-Treasurer, Central High School, Newark, N. J.

Mr. Clyde Lady, of the Swathmore High School, has become a professor of mathematics in the State Normal School at Slippery Rock, Pa.

Mr. Eugene R. Smith, director of the Beaver School, in Boston, and member of the Editorial Board of the Mathematics Teacher, has recently published *Education Moves Ahead,* with the Atlantic Monthly Press. This book is a survey of progressive methods in education.

President Joseph C. Brown, of the St. Cloud Normal School, is the co-author of the Brown-Eldridge Arithmetics which were recently published by Row-Peterson & Co.

Professor W. B. Fite addressed the New York Men Mathematics Teachers Club on "Infinity and Infinitesimal," at the October meeting.